## NOTATION

$\omega$, phase velocity of wave; $\tau$, time of passage of a wave crest from one pickup to the other; $\tau_1$, time of passage of two successive waves over a pickup; H, distance between pairs of electrodes; u, rate of advance of photographic paper of oscillograph; $l$, displacement on oscillogram of wave profiles obtained from different pickups; b, distance between crests of two successive waves on one oscillogram obtained from one pickup; $a_{max}$, $a_{min}$, film thickness at crests and troughs of waves, respectively.

## LITERATURE CITED


1. N. M. Zhavoronkov and V. A. Malyusov, Teor. Osnovy Khim. Tekhnol., 1, No. 5 (1967).
2. S. S. Kutateladze, Principles of Heat and Mass Exchange [in Russian], Nauka, Moscow (1972).
3. H. Breuer, VDI-Forschungsheft, 457 (1956).
4. A. E. Dukler and P. O. Bergelin, Chem. Eng. Progr., 48, No. 11 (1952).
5. F. P. Stainthorp and J. M. Allen, Trans. Inst. Chem. Engrs., 43, No. 3 (1965).
6. R. M. Nedderman and C. J. Shearer, Chem. Eng. Sci., 18, No. 10 (1963).
7. N. H. Taylor, G. F. Hewitt, and P. M. C. Lacey, Chem. Eng. Sci., 18, No. 8 (1963).
8. G. F. Hewitt, R. D. King, and P. C. Lovegrove, AERE-R3921 (1962).
9. J. A. R. Bennett, J. G. Collier, H. R. C. Pratt, and J. D. Thornton, Trans. Inst. Chem. Engrs., 39, No. 1, 113-126 (1961).
10. J. J. van Rossum, Chem. Eng. Sci., 11, No. 1 (1959).
11. A. D. Sergeev and N. A. Nikolaev, Tr. Kazansk. Khim.-Tekhnol. Inst. im. S. M. Kirova, No. 48, 53 (1972).
12. L. P. Kholpanov, G. G. Fedorov, V. A. Malyusov, and N. M. Zhavoronkov, Heat and Mass Transfer, Vol. 4 [in Russian], Minsk (1972), p. 39.
13. N. A. Nikolaev, A. D. Sergeev, L. P. Kholpanov, V. T. Zabrudskii, V. A. Malyusov, and N. M. Zhavoronkov, Teor. Osnovy Khim. Tekhnol., 9, No. 3 (1975).


# LOGARITHMIC EQUATIONS FOR THE RESISTANCE OF TURBULENT FRICTION FOR A VISCOUS LIQUID AND POLYMER SOLUTIONS


V. A. Gorodtsov



UDC 532.517.4



It is shown that resistance equations of the Prandtl type for the flow of a viscous liquid and polymer solutions in the mode of minimal resistance in pipes and near a plate can be approximated with sufficient accuracy by simpler logarithmic equations.


## Flow of a Viscous Liquid in Smooth Pipes

It is well known [1] that in a steady developed turbulent stream near a wall there exists a region of flow with an average velocity distribution close to logarithmic, and in accordance with the asymptotics (as Re → ∞) there exists the logarithmic resistance law (the Prandtl equation)

$$1/\sqrt{\lambda} = a \lg(\mathrm{Re}\sqrt{\lambda}) + b. \tag{1}$$

It turns out that the equation gives satisfactory accuracy in the entire range of Reynolds numbers $4 \cdot 10^3 < \mathrm{Re} < 10^7$ of practical interest if one sets

$$a \approx 2.0; \quad b \approx -0.8. \tag{2}$$


Institute of Problems of Mechanics, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 5, pp. 847-852, November, 1975. Original article submitted June 24, 1974.

A defect of Eq. (1) is the implicit nature of the connection between the resistance coefficient $\lambda$ and the Reynolds number Re. It can be rewritten in the form of an explicit dependence of Re on $\lambda$:

$$t \equiv 10^{b/a}\,\mathrm{Re}/a = y\cdot 10^{y}, \quad y \equiv (a\sqrt{\lambda})^{-1}, \tag{3}$$

although usually, conversely, one has to find the resistance coefficient from the known Reynolds number, i.e., $y = y(t)$.

A circumstance which simplifies the problem is the fact that t is large here (in the range of Reynolds numbers under consideration $8\cdot 10^2 < t < 2\cdot 10^6$ and $2.5 < y < 5.6$).

By rewriting Eq. (3) in the form

$$y + \lg y = \lg t, \tag{4}$$

using successive iterations, we find

$$y = \lg t + O(\lg\lg t) = \lg t - \lg\lg t + O(\lg\lg t/\lg t) = \ldots \tag{5}$$

One can show [2] that in this way one obtains a double series with respect to log t and log log t which converges absolutely at larger t. In principle, this solves the problem of finding the explicit form of the dependence of $\lambda$ on Re which is asymptotically exact with the same condition $\mathrm{Re} \to \infty$ as in the original equation (1).

Simplified equations obtained when only the first-order terms in this expansion are retained are examined below. The relative sizes of the discarded remainders are estimated.

From Eqs. (4) we get the relations

$$y = \lg t + r_1, \quad y = \lg(t/\lg t) - r_2, \ \ldots,$$
$$r_1 = \lg(1/y), \quad r_2 = \lg\left(1 - \frac{r_1}{y}\right), \ \ldots, \ r_{n+1} = \lg(1 - r_n/y). \tag{6}$$

The function $r_1(y)$ varies little (from $-0.4$ to $-0.75$) in the region of Reynolds numbers under consideration, and by replacing it with the linear function $c_1y + d_1$ we arrive at the resistance equation

$$1/\sqrt{\lambda} \approx a'\lg\mathrm{Re} + b', \tag{7}$$

$$a' = \frac{a}{1-c_1}, \quad b'/a' = b/a + d_1 - \lg a. \tag{8}$$

By choosing the coefficients $c_1$ and $d_1$ of the linear approximation so that the error in the determination of y is minimal in the range of its variation under consideration, we find, using (2), that

$$a' \approx 1.79; \quad b' \approx -1.46. \tag{9}$$

Here the difference in $\lambda$ calculated from Eqs. (7) and (1) reaches only 1% at the ends of the range under consideration and at some middle point $\mathrm{Re} \approx 6\cdot 10^5$.

An equation of the type of (7) has been used repeatedly in work on hydraulics [3]. It was found that within the limits of the experimental errors the predictions of the simpler equation (7) and the Prandtl equation coincide and are satisfactory. Attempts at the theoretical derivation of Eq. (7) independent of the derivation of (1) which have been made earlier have turned out, as is known [4, 5], to be unsound.

The residual term $r_2$ varies from 0.055 to 0.064, i.e., within even smaller limits than $r_1(y)$. Approximating $r_2$ by the constant 0.06, we obtain a solution of Eq. (1) which is no less accurate than in the preceding case:

$$0.5/\sqrt{\lambda} \approx \lg(0.2\,\mathrm{Re}) - \lg\lg(0.2\,\mathrm{Re}) + 0.06. \tag{10}$$

It is not hard to write a whole series of such equations which are obtained through the replacement of the functions $y - r_n(y)$ by linear functions which are close to them in the range of y under consideration. One can also obtain equations of a somewhat different type (more accurate, in particular) by approximating $y - r_n(y)$ with other functions which have simple inverse functions.

By replacing y + log y with the power function $cy^{1/n} + d$ we obtain the following resistance equation in place of (1):

$$\lambda \approx (a'' \lg \mathrm{Re} + b'')^{-2n}, \quad (11)$$

$$a'' = a^{1/n}/c, \quad b''/a'' = b/a - d - \lg a. \quad (12)$$

By choosing the coefficients c, d, and n so that the equality $y + \log y = cy^{1/n} + d$ is satisfied at the end points $y_1 = 2.5$ and $y_2 = 5.56$ and the middle point $\sqrt{y_1 y_2}$ we obtain

$$\lambda \approx (1.31 \lg \mathrm{Re} - 0.5)^{-2.22}. \quad (13)$$

Since the inaccuracy of the approximation used here for the function y + log y is less than 1% in the indicated range, Eq. (13) is a good approximation for the dependence (1).

The other equations can also be simplified using the simplified resistance equations (7) and (11). For example, in place of the expression for the average velocity distribution (5.6 corresponds to $a$ = 2.0)

$$\langle u^+ \rangle = 5.6 \lg z^+ + B_0 \quad (30 < z^+ < 0.2\, r^+) \quad (14)$$

one can use (7) and (9) to write

$$\langle u^+ \rangle \approx 5.0 \lg \mathrm{Re} + B_0 - 6.1 + 5.6 \lg (z/r), \quad (15)$$

while for the maximum velocity we obtain

$$U^+ \approx 5.0 \lg \mathrm{Re} + B_0 + B_1 - 6.1. \quad (16)$$

Equations of a similar type are used in hydraulics [3].

Turbulent Flow of a Viscous Liquid over a Plate

The equations for the resistance of a smooth flat plate over which a pressureless stream of viscous liquid flows have a form similar to Eq. (1), as is known [1, 6].

For the local resistance coefficient

$$1/\sqrt{c_f} = \bar{a} \lg (\mathrm{Re}_x c_f) + \bar{b}, \quad (17)$$

following the change

$$1/\sqrt{c_f} = 2\bar{a}y, \quad \mathrm{Re}_x = (2\bar{a}t)^2\, 10^{-\bar{b}/\bar{a}}, \quad (18)$$

we obtain Eq. (4) as before. If in accordance with the experimental measurements of [1] we take

$$\bar{a} = 4.15; \quad \bar{b} = 1.7, \quad (19)$$

then in the range of Reynolds numbers of practical interest ($5 \cdot 10^5 \leqslant \mathrm{Re}_x \leqslant 5 \cdot 10^9$) the parameter t will be almost exactly as large as for flow in pipes ($1.35 \cdot 10^2 < t < 1.35 \cdot 10^4$ and $1.86 < y < 3.58$). In complete analogy with the foregoing, therefore, we will have as the approximate solutions of Eq. (17)

$$1/\sqrt{c_f} \approx 3.55 \lg \mathrm{Re}_x - 4.8, \quad (20)$$

$$0.12/\sqrt{c_f} \approx u - \lg u + 0.06, \quad u \approx \lg (0.193 \sqrt{\mathrm{Re}_x}), \quad (21)$$

$$c_f \approx (2 \lg \mathrm{Re}_x - 0.8)^{-2.32}. \quad (22)$$

All these expressions convey the dependence $c_f(\mathrm{Re}_x)$ described by Eq. (17) with an accuracy no worse than 1%.

For the total resistance coefficient $C_f$ one can use in place of the equation

$$1/\sqrt{C_f} = 4.13 \lg (\mathrm{Re}_L C_f) \quad (23)$$

the approximate relations

$$1/\sqrt{C_f} \approx 3.56 \lg \mathrm{Re}_L - 6.0, \quad (24)$$

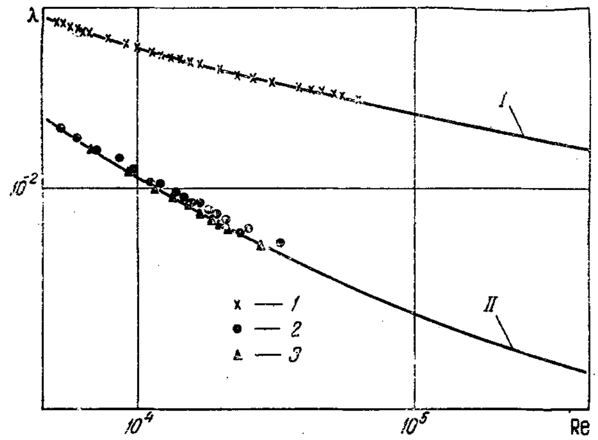


Fig. 1. Resistance of turbulent friction as a function of the Reynolds number. Curves I and II are constructed from Eqs. (13) and (29). Points 1, 2, and 3 pertain to water and to a polymer solution which is at different distances from the entrance to the pipe, greater than 370r and 550r, respectively (see Fig. 3 in [9]).

$$0.121/\sqrt{C_f} \approx u - \lg u + 0.06, \quad u \approx -0.92 + \lg \sqrt{\mathrm{Re}_L}, \tag{25}$$

$$C_f \approx (2.12 \lg \mathrm{Re}_L - 1.94)^{-2.28}. \tag{26}$$

Equations (22) and (26) have received wide distribution in the literature [1, 6].

Minimum Resistance of Polymer Solutions in Pipes

Under certain limiting conditions of the flow of polymer solutions of low concentration in smooth pipes [7] the coefficient of resistance of turbulent friction proves to be independent of the specific type of medium and of the pipe diameter, just as in the flow of viscous liquids. Under such conditions the Prandtl equation (1) is valid as before, and according to the experimental data of [8, 9]

$$a \approx 9.5; \quad b \approx -19.1. \tag{27}$$

In this case the equation can be reduced to the form of (4) using the substitution (3). Now, however, the parameter is not as large as before. With variation in the Reynolds number from $5 \cdot 10^3$ to $10^6$ we have $5.1 < t < 10^3$ $(0.8 < y < 2.6)$. Therefore, the accuracy of the approximations under consideration proves to be lower, despite the narrower range of variation of Re.

With the same values of the constants as in (27) the linear approximation of the function $y + \log y$ leads to the equation

$$1/\sqrt{\lambda} \approx 7.25 \lg \mathrm{Re} - 19.3. \tag{28}$$

This equation conveys the dependence (1), (27) with an accuracy of several percent [the difference in the determination of λ from (28) and from (1), (27), can reach 5% in the range of Reynolds numbers under consideration].*

As before, a power-law approximation of the function $y + \log y$ has good accuracy. It leads to the equation

$$\lambda \approx (3.15 \lg \mathrm{Re} - 6.8)^{-2.58}, \tag{29}$$

which conveys the dependence (1), (27) with 1% accuracy. In Fig. 1 curves are constructed on the basis of (13) and (29) and experimental data are plotted for water and for an aqueous

*Of course, the accuracy of the approximation can be improved if one considers a narrower range of Reynolds numbers of higher ones.

solution of polyoxyethylene with a concentration of $10^{-5}$, taken from [9]. The points 2, corresponding to measurements of the resistance at a smaller distance from the entrance to the pipe, lie somewhat above the curve based on (29), while those for a greater distance (3) lie below.†

## Flow of Polymer Solutions in Pipes (General Case)

As was established in [10], for polymer solutions of low concentrations the dependence of the resistance to flow in pipes is also described satisfactorily by the Prandtl equation with coefficients which depend on the properties of the solution (b also depends on the pipe diameter):

$$\Delta a \equiv a - 2 = \alpha/\sqrt{8},$$
$$\Delta b \equiv b + 0.8 = -\Delta a \lg (5.66 u_{*cr} r/\nu). \tag{30}$$

Here $\alpha$ is a dimensionless parameter [7, 10] and $u_{*cr}$ is the value of the dynamic velocity which characterizes the start of anomalous resistance ($\Delta a = 0$ for $u_* < u_{*cr}$).

As was noted, the Prandtl equation is reduced to (4) by the substitution (3). The approximations discussed earlier prove to be less accurate in this case, however, since the parameter now can vary within much wider limits (according to [7] the maximum value of $a$ is close to 17, i.e., larger than in (27)), while $\lambda$ can be larger than $2 \cdot 10^{-2}$ for small Reynolds numbers. This leads to the fact that y varies within wider limits, while t can take on values less than unity.

If we use a power-law approximation of the function y + log y with constant coefficients n, c, and d, then in accordance with (11), (12), and (30) we obtain

$$\lambda \approx c^{2n}/a^2 \{\lg [\mathrm{Re}\,(5.66 u_{*cr} r/\nu)^{2/a-1}] - d - \lg a - 0.8/a\}^{-2n}. \tag{31}$$

The difference in $\lambda$ calculated from this equation and from an equation of the Prandtl type can even reach tens of percent. The accuracy could be increased by selecting coefficients n, c, and d which depend on the characteristics of the solution and the pipe diameter. Very cumbersome equations are obtained in this case, however.

The author thanks V. S. Belokon', V. N. Kalashnikov, and B. V. Lipatov for a discussion and for the presentation of the data of experimental measurements.

## NOTATION

r, pipe radius; z, distance from wall; <u>, average flow velocity; U, maximum velocity; $z^+ \equiv zu_*/\nu$, $u^+ \equiv u/u_*$, dimensionless wall variables; $\nu$, viscosity of liquid; $u_*$, dynamic velocity; $c_f$, local resistance coefficient; $C_f$, total resistance coefficient.

## LITERATURE CITED

1. A. S. Monin and A. M. Yaglom, Statistical Fluid Mechanics, Part 1 [in Russian], Nauka, Moscow (1966).
2. N. G. De Bruijn, Asymptotic Methods in Analysis, Wiley-Interscience, New York (1958).
3. A. D. Al'tshul', Hydraulic Resistance [in Russian], Nedra, Moscow (1970).
4. A. N. Kolmogorov, Dokl. Akad. Nauk SSSR, 52, 669 (1946).
5. A. N. Kolmogorov, Dokl. Akad. Nauk SSSR, 84, 29 (1952).
6. H. Schlichting, Boundary Layer Theory
7. V. A. Gorodtsov and V. S. Belokon', Inzh.-Fiz. Zh., 25, 967 (1973).
8. P. S. Virk, H. S. Mickley, and K. A. Smith, Trans. ASME, E37, 488 (1970); Prikl. Mekh., No. 2, 238 (1970).
9. V. S. Belokon', V. N. Kalashnikov, and B. V. Lipatov, Inzh.-Fiz. Zh., 25, 1010 (1973).
10. W. A. Meyer, Amer. Inst. Chem. Eng. J., 12, 522 (1966).

†Points which depart strongly from the curve at larger Reynolds numbers are not presented here since they evidently pertain to a strongly degraded solution.